Functions of two variables

Young's Theorem :→

Statement :→ Let f be a scalar field defined in some neighbourhood of a point $(a,b) \in \mathbb{R}^2$ s.t.

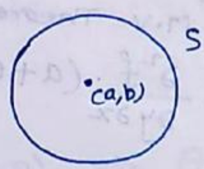
(i) both the partial order derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exists finitely in some neighbourhood of the point (a,b).

(ii) both the partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exists and are continuous in some neighbourhood of the point (a,b),

Then

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

Proof :→



∵ The given conditions are ~~satisfies~~ satisfied in some neighbourhood S of the point (a,b), We can always find $h, k \in \mathbb{R}$ s.t. the points $(a+h, b)$, $(a, b+k)$, $(a+h, b+k)$ lie entirely in S.

Let

$$\Delta(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$$

Let us define

$$\phi(x) = f(x, b+k) - f(x, b)$$

for all $x \in [a, a+h]$ or $[a+h, a]$ depending on whether $h>0$ or $h<0$. So,

$$\Delta(h,k) = \phi(a+h) - \phi(a)$$

$\because$ $\frac{\partial f}{\partial x}$ exists and is continuous in some neighbourhood $\cdot$ S, ϕ' exists in S.

Applying Mean value theorem.

$$\Delta(h,k) = \phi(a+h) - \phi(a)$$

$$= h\,\phi'(a+\theta_1 h) \quad \text{Where } 0<\theta_1<1.$$

$$= h\left\{\frac{\partial f}{\partial x}(a+\theta_1 h, b+k) - \frac{\partial f}{\partial x}(a+\theta_1 h, b)\right\}$$

Applying M.V. Theorem

$$= h\left\{k\,\frac{\partial^2 f}{\partial y\,\partial x}(a+\theta_1 h, b+\theta_2 k)\right\} \quad \text{--- (1)}$$

Where $0<\theta_2<1$

Obviously $(a+\theta_1 h, b+\theta_2 k)$ lies inside the rectangle with (a,b) and $(a+h, b+k)$ as digonal points.

Let us define

$$\psi(x) = f(a+h, y) - f(a, y)$$

for all $x \in [b, b+k]$ or $[b+k, b]$ depending on whether $k>0$ or $k<0$. So using the similar argument as above, we obtain.

$$\Delta(h,k) = \psi(b+k) - \psi(b)$$

$$= k\,\psi'(b+\theta_3 h), \text{ where } 0<\theta_3<1.$$

$$= k\left\{\frac{\partial f}{\partial y}(a+h, b+\theta_3 k) - \frac{\partial f}{\partial y}(a, b+\theta_3 k)\right\}$$

Applying M.V. Theorem

$$= k\left\{h\,\frac{\partial^2 f}{\partial x \partial y}(a+\theta_4 h, b+\theta_3 k)\right\} \quad \text{— (2)}$$

where $0<\theta_4<1$.

Here also the point $(a+\theta_4 h, b+\theta_3 k)$ lies inside the rectangle with (a,b) and $(a+h, b+k)$ as digonal points.

Clearly

$$(h,k) \to (0,0) \Rightarrow \theta_1, \theta_2, \theta_3, \theta_4 \to 0$$

from expression (1) & (2), we obtain the following;

$$hk\left\{\frac{\partial^2 f}{\partial y \partial x}(a+\theta_1 h, b+\theta_2 k)\right\} = hk\left\{\frac{\partial^2 f}{\partial x \partial y}(a+\theta_4 h, b+\theta_3 k)\right\}$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x}(a+\theta_1 h, b+\theta_2 k) = \frac{\partial^2 f}{\partial x \partial y}(a+\theta_4 h, b+\theta_3 k)$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} \frac{\partial^2 f}{\partial y \partial x}(a+\theta_1 h, b+\theta_2 k) = \lim_{(h,k)\to(0,0)} \frac{\partial^2 f}{\partial x \partial y}(a+\theta_4 h, b+\theta_3 k)$$

Taking limit of both sides of the above relation as $(h,k) \to (0,0)$ we find

$$\frac{\partial^2 f}{\partial y \partial x}(a,b) = \frac{\partial^2 f}{\partial x \partial y}(a,b)$$

$\because$ Both $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are continuous at (a,b).

This complets the proof.